

Notes on Approximation Algorithms for Polynomial-Expansion and Low-Density Graphs*

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Abstract

This write-up contains some minor results and notes related to our work [HQ15]. In particular, it shows the following:

- (A) In [Section 1](#) we show that a graph with polynomial expansion have sublinear separators.
- (B) In [Section 2](#) we show that hereditary sublinear separators imply that a graph have small divisions.
- (C) In [Section 3](#), we show a natural condition on a set of segments, such that they have low density. This might be of independent interest in trying to define a realistic input model for a set of segments. Unlike the previous two results, this is new.

For context and more details, see the main paper [HQ15].

1. Polynomial expansion implies sublinear separators

Definition 1.1. Let $G = (V, E)$ be an undirected graph. Two sets $X, Y \subseteq V$ are *separate* in G if (i) X and Y are disjoint, and (ii) there is no edge between the vertices of X and Y in G . A set $Z \subseteq V$ is a *separator* for a set $U \subseteq V$, if $|Z| = o(|U|)$, and $U \setminus Z$ can be partitioned into two *separate* sets X and Y , with $|X| \leq (2/3)|V|$ and $|Y| \leq (2/3)|V|$ ¹.

Theorem 1.2 ([PRS94, Theorem 2.3]). *Let G be a graph with m edges and n vertices, and let $\ell, h \in \mathbb{N}$ be two integer parameters. There is an $O(mn/\ell)$ time algorithm that either produces*

- (a) *the clique K_h as a $\ell \log n$ -shallow minor of G , or*
- (b) *a separator of size at most $O(n/\ell + 4\ell h^2 \log n)$.*

Theorem 1.3 ([NO08, Theorem 8.3]). *Let \mathcal{C} be a class of graphs with polynomial expansion of order k . For any graph $G \in \mathcal{C}$ with n vertices and m edges, one can compute, in $O(mn^{1-\alpha} \log^{1-\alpha} n)$ time, a separator of size $O(n^{1-\alpha} \log^{1-\alpha} n)$, where $\alpha = 1/(2k + 2)$.*

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¹Here, the choice of $2/3$ is arbitrary, and any constant smaller than 1 is sufficient.

Proof: Let z be a parameter to be fixed shortly, and let $\ell = z/\log n$ and $cz^k/4 > \mathfrak{d}_z(G)$, where c is a sufficiently large constant. Consider a z -shallow minor H of G with $h = cz^k$ vertices, and observe that by definition, we have that $|E(H)| \leq \mathfrak{d}_z(G) |V(H)| < \frac{cz^k}{4} cz^k < \binom{h}{2}$. That is, the graph H can not be the clique K_h .

Now, by [Theorem 1.2](#), G has a separator of size

$$O\left(n/\ell + \ell h^2 \log n\right) = O\left(\frac{n \log n}{z} + \frac{z}{\log n} \cdot z^{2k} \cdot \log n\right) = O\left(\frac{n \log n}{z} + z^{2k+1}\right) = O\left(n^{\frac{2k+1}{2k+2}} \log^{\frac{2k+1}{2k+2}} n\right)$$

for $z = n^{1/(2k+2)} \log^{1/(2k+2)} n$. The algorithm provided by [Theorem 1.2](#) runs in time $O\left(\frac{mn}{\ell}\right) = O\left(\frac{mn \log n}{z}\right) = O\left(mn^{\frac{2k+1}{2k+2}} \log^{\frac{2k+1}{2k+2}} n\right)$. \blacksquare

2. Hereditary separators imply small divisions

Consider a set V . A **cover** of V is a set $\mathcal{W} = \{C_i \subseteq V \mid i = 1, \dots, k\}$ such that $V = \bigcup_{i=1}^k C_i$. A set $C_i \in \mathcal{W}$ is a **cluster**. A cover of a graph $G = (V, E)$ is a cover of its vertices. Given a cover \mathcal{W} , the **excess** of a vertex $v \in V$ that appears in j clusters is $j - 1$. The **total excess** of the cover \mathcal{W} is the sum of excesses over all vertices in V .

Definition 2.1. A cover \mathcal{C} of G is a **λ -division** if (i) for any two clusters $C, C' \in \mathcal{C}$, the sets $C \setminus C'$ and $C' \setminus C$ are separated in G (i.e., there is no edge between these sets of vertices in G), and (ii) for all clusters $C \in \mathcal{C}$, we have $|C| \leq \lambda$.

A vertex $v \in V$ is an **interior vertex** of a cover \mathcal{W} if it appears in exactly one cluster of \mathcal{W} (and its excess is zero), and a **boundary vertex** otherwise. By property (i), the entire neighborhood of an interior vertex of a division lies in the same cluster.

The property of having λ -divisions is slightly stronger than being weakly hyperfinite. Specifically, a graph is *weakly hyperfinite* if there is a small subset of vertices whose removal leaves small connected components [[NO12](#), Section 16.2]. Clearly, λ -divisions also provide such a set (i.e., the boundary vertices). The connected components induced by removing the boundary vertices are not only small, but the neighborhoods of these components are small as well.

As noted by Henzinger *et al.* [[Hen+97](#)], strongly sublinear separators obtain λ -divisions with total excess εn for $\lambda = \text{poly}(1/\varepsilon)$. Such divisions were first used by Frederickson in planar graphs [[Fre87](#)].

Lemma 2.2 ([[Hen+97](#)]). *Let G be a graph with n vertices, such that any induced subgraph with m vertices has a separator with $O(m^\alpha \log^\beta m)$ vertices, for some $\alpha < 1$ and $\beta \geq 0$. Then, for $\varepsilon > 0$, the graph G has λ -divisions with total excess εn , where $\lambda = O\left((\varepsilon^{-1} \log^\beta \varepsilon^{-1})^{1/(1-\alpha)}\right)$.*

Proof: Our strategy is to break G into smaller pieces. Specifically, at every step the algorithm takes the largest remaining piece $G|_U$, compute a balanced separator $Z \subseteq U$ for it, with $L, R \subseteq U$ being the two separated pieces. Specifically, we have

- (i) $Z = L \cap R$,
- (ii) $L \cup R = U$,
- (iii) $|L| \leq (2/3) |U|$ and $|R| \leq (2/3) |U|$ (see [Definition 1.1](#)),
- (iv) $L \setminus Z$ is separated from $R \setminus Z$ in $G|_U$, and
- (v) $|Z| \leq f(|U|)$, where $f(m) \leq cm^\alpha \log^\beta m$, where c is a sufficiently large constant.

Now, the algorithm replaces G_U by the two “broken” pieces G_L and G_R . The algorithm continues in this process until all pieces are of size smaller than b (and by construction, of size at least, say, $b/4$), where b is some parameter to be specified shortly. This generates a natural binary separator tree, where the final pieces of the division are the leafs.

Let $N_i = (3/4)^i n$, for $i = 0, \dots, h = \lceil \log_{4/3} n \rceil$. A piece G_U is at *level* i if $N_{i+1} < |U| \leq N_i$. Consider such a subproblem at node y , which is at level i with ν vertices. The total size of the subproblems of its two children is $\leq \nu + 2f(\nu)$ (here, somewhat confusingly, we count the separator vertices as new, in both subproblems – this makes the following argument somewhat easier). Importantly, each of the subproblems is of size $\leq (2/3)\nu + f(\nu) \leq (3/4)\nu$, implying that both subproblems are in strictly lower level. As such, the fraction of the new vertices created as subproblems move from the i th level to the next is bounded by

$$\nu + 2f(\nu) \leq \nu + 2c\nu^\alpha \log^\beta \nu = \left(1 + \frac{2c \log^\beta \nu}{\nu^{1-\alpha}}\right) \nu \leq \gamma_i \nu,$$

for $\gamma_i = 1 + 2 \frac{c \log^\beta N_{i+1}}{(N_{i+1})^{1-\alpha}}$. In particular, the total number of vertices in the k th level is at most $\Delta_k n$, where

$$\begin{aligned} \Delta_k &= \prod_{j=0}^{k-1} \gamma_j \leq \prod_{j=0}^{k-1} \exp\left(2 \frac{c \log^\beta N_{j+1}}{(N_{j+1})^{1-\alpha}}\right) = \exp\left(\sum_{j=0}^{k-1} \frac{2c \log^\beta N_{j+1}}{(N_{j+1})^{1-\alpha}}\right) \leq \exp\left(\frac{c' \log^\beta N_k}{(N_k)^{1-\alpha}}\right) \\ &\leq 1 + \frac{2c' \log^\beta N_k}{(N_k)^{1-\alpha}}, \end{aligned}$$

since the summation behaves like an increasing geometric series, and c' is a constant that depends on c . The last step follows as $e^x \leq 1 + 2x$, for $0 \leq x \leq 1/2$. In particular, because of the double counting of the separator vertices, the total number of marked vertices in the first k levels is bounded by $n(\Delta_k - 1)$. As such, we need that $\Delta_k - 1 \leq \varepsilon$. This is equivalent to

$$\frac{2c' \log^\beta N_k}{(N_k)^{1-\alpha}} \leq \varepsilon \iff \frac{2c'}{\varepsilon} \leq \frac{(N_k)^{1-\alpha}}{\log^\beta N_k},$$

which holds if $N_k \geq (c'' \varepsilon^{-1} \log^\beta \varepsilon^{-1})^{1/(1-\alpha)}$, where c'' is a sufficiently large constant. In particular, setting b to (say) twice this threshold implies the claim. \blacksquare

3. On exposed sets of segments and their density

Let $\sigma > 0$ be a fixed parameter. We say that an object f σ -*shadows* (or simply *shadows*) another object g if

$$\max_{q \in g} d(q, f) \leq \sigma \cdot \text{diam}(g),$$

where $d(q, f) = \min_{u \in f} \|q - u\|$. Equivalently, f σ -shadows $g \iff g \subseteq f \oplus \mathbb{b}(0, \sigma \cdot \text{diam}(g))$. Here, $X \oplus Y = \{q + u \mid q \in X, u \in Y\}$ denotes the *Minkowski sum* of X and Y . A set of objects \mathcal{U} is σ -*exposed* if no object in \mathcal{U} σ -shadows another object in \mathcal{U} .

Observation 3.1. *Let f and g be two objects and $\sigma \geq 0$. If $f \subseteq g$, then g σ -shadows f .*

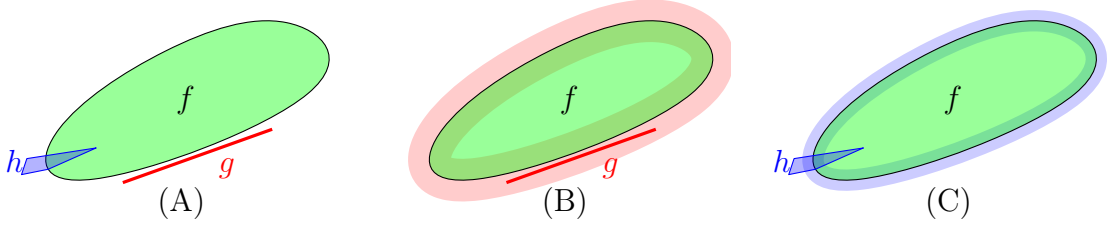


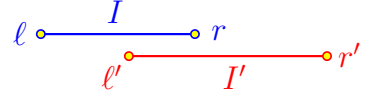
Figure 3.1: (A) Objects f, g, h . (B) f 1/8-shadows g . (C) f does not 1/8-shadow h .

3.1. On the density of exposed segments

3.1.1. Intervals in \mathbb{R}

Following the above, interval $I = [\ell, r]$ **σ -exposes** $I' = [\ell', r']$, if I' is not contained in the interval $[\ell - \sigma \|I\|, r + \sigma \|I\|]$, where $\|I'\| = \ell' - r'$ denotes the length of I' .

Lemma 3.2. *Let $I = [\ell, r]$ and $I' = [\ell', r']$ be two overlapping intervals on the real line. If I and I' σ -expose each other, then $|\ell' - \ell| \geq \sigma \|I\|$ and $|r' - r| \geq \sigma \|I'\|$.*



Proof: Without loss of generality, assume that $\ell \leq \ell'$. Since I and I' are overlapping, we have $\ell' \leq r$. Furthermore, if $r' \leq r$, then $I \subseteq I'$ and by **Observation 3.1** the interval I σ -shadows I' . So it must be that $r' > r$. Since the left endpoint ℓ' of I' is contained in I , I does not σ -shadow I' only if r' extends at least $\sigma \|I'\|$ past r . Similarly, if I' does not σ -shadow I , then $\ell' - \ell \geq \sigma \|I\|$. ■

Lemma 3.3. *Let \mathcal{I} be a set of intervals all covering a common point p . If \mathcal{I} is a σ -exposed set of intervals, then $|\mathcal{I}| = O(1/\sigma^2)$.*

Proof: Let the i th interval of \mathcal{I} be $I_i = [\ell_i, r_i]$, for $i = 1, \dots, n$. Furthermore, assume that $\ell_1 \leq \ell_2 \leq \dots \leq \ell_n$. By Dilworth's theorem, there exists a subsequence $i_1 < i_2 < \dots < i_k$ with $k \geq \sqrt{n}$, such that either $r_{i_1} \leq r_{i_2} \leq \dots \leq r_{i_k}$ or $r_{i_1} \geq r_{i_2} \geq \dots \geq r_{i_k}$. The later possibility implies that $I_{i_1} \subseteq I_{i_2}$, which contradicts the assumption that \mathcal{I} is σ -exposed. Assume, without loss of generality, that for at least half the intervals in this sequence, we have $|\ell(I)| \geq r(I)$, and let $I'_1, \dots, I'_{k/2}$ be the resulting subsequence restricted to these intervals, where $I'_i = [\ell'_i, r'_i]$ for all i . (The other case is handled by symmetric argument.)

We have $\ell'_1 \leq \dots \leq \ell'_{k/2} \leq 0 \leq r'_1 \leq \dots \leq r'_{k/2}$. By **Lemma 3.2**, we have that for any i , we have $r'_i - r'_{i-1} \geq \sigma \|I'_{i-1}\| \geq \sigma |\ell'_{i-1}|$. Summing this inequality for $i = 2, \dots, t$, we have

$$r'_t > r'_t - r'_1 \geq \sigma \sum_{i=1}^{t-1} |\ell'_i| \geq \sigma \cdot (t-1) |\ell_t| > |\ell_t|,$$

for $t = \lceil 1/\sigma \rceil + 2$, which is a contradiction. We conclude that $\sqrt{n}/2 \leq k/2 \leq \lceil 1/\sigma \rceil + 2$, which readily implies the claim. ■

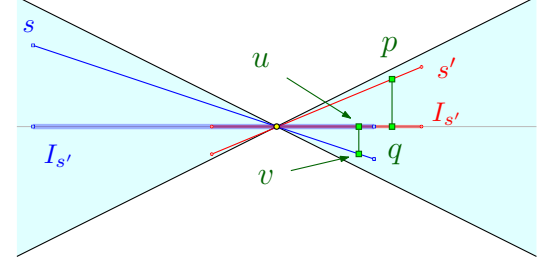
3.1.2. Line segments through a point

Lemma 3.4. *Let \mathcal{L} be a set of segments in \mathbb{R}^d , and $\sigma > 0$, $\theta \in (0, \pi/2)$ be parameters. Furthermore, assume that (i) \mathcal{L} is σ -exposed, (ii) $\bigcap_{s \in \mathcal{L}} s \neq \emptyset$, (iii) for all pairs $\ell_1, \ell_2 \in \mathcal{L}$, the angle between ℓ_1 and ℓ_2 is at most θ , and (iv) $\sin \theta \leq \frac{\sigma}{4}$. Then $|\mathcal{L}| = O(1/\sigma^2)$.*

Proof: Without loss of generality, we assume that the lines intersect at the origin and the angle between any line and the x -axis is at most θ . For each $s \in \mathcal{L}$, let $\ell(s)$ be the left endpoint of s , let $x_\ell(s)$ be the x -coordinate of $\ell(s)$, and let $y_\ell(s)$ the distance from $\ell(s)$ to the x -axis. Similarly we define $r(s)$, $x_r(s)$, and $y_r(s)$ with respect to the right endpoint. For $s \in \mathcal{L}$, let $I_s = [x_\ell(s), x_r(s)]$ be the projection of s onto the x -axis, and let $\mathcal{I}_\mathcal{L} = \{I_s \mid s \in \mathcal{L}\}$.

We claim that \mathcal{I} is $(\sigma/4)$ -exposed. Indeed, suppose that there are two segments s, s' , such that $I_s = [\ell, r]$ is $\sigma/4$ -shadowing $I_{s'} = [\ell', r']$. We define the following sequence of points:

- (i) p is any point on s' ,
- (ii) q is the projection of p into $I_{s'}$,
- (iii) u is any point in I_s that is in distance at most $(\sigma/4) \|I_{s'}\|$ from q , and it is closer to the origin than u , and
- (iv) v is the point on s whose projection on I_s is u .



See the figure on the right.

We now have that $\|p - q\| \leq \|s'\| \sin \theta \leq (\sigma/4) \|s'\|$. Similarly, as u is closer to the origin than q , we have that $\|u - v\| \leq (\sigma/4) \|s'\|$. Also, since $\|q - u\| \leq (\sigma/4) \|I_{s'}\| \leq (\sigma/4) \|s'\|$, we have by the triangle inequality that $\|p - v\| \leq \|p - q\| + \|q - u\| + \|u - v\| \leq \sigma \|s'\|$, which implies that s' is σ -shadowed by s , a contradiction.

Now, **Lemma 3.3** implies that $|\mathcal{L}| = |\mathcal{I}_\mathcal{L}| = O(1/\sigma^2)$. ■

Lemma 3.5. *Let \mathcal{L} be a set of segments in \mathbb{R}^d and $\sigma \in (0, 1)$ a fixed parameter, such that (i) \mathcal{L} is σ -exposed, and (ii) $\bigcap_{s \in \mathcal{L}} s \neq \emptyset$. Then $|\mathcal{L}| = O(1/\sigma^{d+2})$.*

Proof: Partition \mathcal{L} into $O(\sigma^{-d})$ clusters such that any two lines in the same cluster forms an angle $\leq \sigma/4$. By **Lemma 3.4**, each cluster contains at most $O(1/\sigma^2)$ segments, and the claim follows. ■

3.1.3. Large segments all intersecting a common ball

Lemma 3.6. *Let b be a ball of radius r , and let \mathcal{L} be a set of segments both in \mathbb{R}^d . Furthermore, assume that (i) \mathcal{L} is σ -exposed, (ii) all the segments of \mathcal{L} intersect b , and (iii) they are all of length $\geq r$. Then, we have $|\mathcal{L}| = O(1/\sigma^{2d+2})$.*

Proof: Let \mathcal{B} be a set of $O(\sigma^{-e})$ balls of radius $\sigma r/4$, that cover b . For each $s \in \mathcal{L}$, pick a small ball $b_s \in \mathcal{B}$ intersecting s , and translate s by at most $\sigma r/4$ so that it passes through the center of b_s . For $s \in \mathcal{L}$, let s' denote the translated segment, and let $\mathcal{L}' = \{s' \mid s \in \mathcal{L}\}$.

Since \mathcal{L} is σ -exposed, and the length of each segment of \mathcal{L} is at least r , it follows that \mathcal{L}' is $\sigma/2$ -exposed, as can be easily verified.

Now, for every ball $b(c, \sigma r/4) \in \mathcal{B}$, consider the segment of segments $\mathcal{L}'(c)$ that passes through c . By **Lemma 3.5**, we have $|\mathcal{L}'(c)| = O(1/\sigma^{d+2})$. This implies that $|\mathcal{L}| = |\mathcal{L}'| = O(|\mathcal{B}|/\sigma^{d+2}) = O(1/\sigma^{2d+2})$. ■

3.1.4. Putting things together

Lemma 3.7. *Let \mathcal{L} be a set of segments in \mathbb{R}^d and $\sigma > 0$ a fixed parameter. If \mathcal{L} is σ -exposed, then \mathcal{L} has density $O(\sigma^{-2d-2})$.*

Proof: Consider any ball $b(c, r)$ in \mathbb{R}^d . By **Lemma 3.6**, there could be at most $O(\sigma^{-2d-2})$ segments of length $\geq 2r$ of \mathcal{L} intersecting it, and the result follows. ■

3.2. On (σ, k) -shadowing

A set of objects \mathcal{U} in \mathbb{R}^d is (σ, k) -*exposed* if each object $f \in \mathcal{U}$ is σ -shadowed by at most k other objects in \mathcal{U} .

Lemma 3.8. *Let $\sigma > 0$ be a fixed parameter and \mathcal{V} a set of objects, such that for any subset $\mathcal{H} \subseteq \mathcal{V}$ that is σ -exposed, we have that $\text{density}(\mathcal{H}) \leq \rho$. If \mathcal{V} is (σ, k) -exposed, then $\text{density}(\mathcal{V}) \leq (2k + 1)\rho$.*

Proof: We create a graph G over \mathcal{V} , with an edge between two objects $g, h \in \mathcal{V}$ if one shadows the other. By assumption, the average degree in G is bounded by $2k$, and in particular the graph is $2k$ -degenerate and can be partitioned into $2k + 1$ independent sets. Every independent set is σ -exposed, and by assumption has density $\leq \rho$. Since density is subadditive under unions, \mathcal{V} has density at most $(2k + 1)\rho$. ■

Corollary 3.9. *Let \mathcal{L} be a set of segments in \mathbb{R}^d that (σ, k) -exposed. Then \mathcal{L} has density $O(k\sigma^{-4})$.*

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